JOURNAL OF COMPUTATIONAL PHYSICS 10, 475–486 (1972)

Nonphysical Noises and Instabilities in Plasma Simulation due to a Spatial Grid

HIDEO OKUDA

Plasma Physics Laboratory, Princeton University, Princeton, New Jersey 08540

Received January 10, 1972

A series of plasma numerical simulation has been performed in order to understand the enhancement of nonphysical noises and instabilities due to the use of a spatial grid. Several different superparticle models including the Nearest Grid Point (NGP) model, Cloud-in-Cell (CIC) or Particle-in-Cell (PIC) models, Lewis energy conserving code, and the multipole expansion code have been examined for a Maxwellian plasma and a one beam plasma using a one-dimensional, one-specie (electron) plasma. An instability was observed for all of the models when the Debye length was too small compared with the grid size. When the Debye length is comparable to the grid size, no instabilities were observed. However, the enhancement of noises at high frequencies ($\omega \gtrsim 3\omega_{pe}$) may not always be negligible even for long wavelength modes for the NGP model. For the NGP and CIC, PIC models, the experimental results are in good agreement with Langdon's theory. It is observed that the dipole expansion model, which is the first-order approximation to the multipole expansion scheme, is similar to CIC, PIC models in many respects and appears to be the same order of approximation.

INTRODUCTION

This paper presents a numerical study on the nonphysical noises and instabilities arising from the use of a spatial grid in various superparticle models now in use in plasma simulation. The models examined include the Nearest Grid Point (NGP) model [1], Cloud-in-Cell (CIC) [2] or Particle-in-Cell (PIC) [3] models, Lewis energy conserving code [4], and the multipole expansion code [5]. Since the use of a spatial grid in plasma simulation with particles is now widespread, and there is a general inclination to economize, using coarse time and space grids, it is important to understand the effects of such grid noises, especially in the enhancement of nonphysical noise or instabilities in order to distinguish physical information from nonphysical information.

The presence of grid instabilities was predicted by Lindman [6] and by Langdon [7]. Langdon predicts that when the Debye length is small compared with the grid size, numerical modes due to the discreteness of a spatial grid, called aliases, can destabilize plasma oscillations, even in a thermal plasma. When the Debye length is comparable to the grid size, no instabilities will be observed. However, the enhancement of the noises at high frequencies may not always be small.

With a one-specie (electron), one-dimensional plasma, NGP, CIC, PIC, Lewis energy conserving code and the dipole expansion scheme have been examined for both Maxwellian and beam plasmas. The properties of various models are discussed from different points of view, such as the energy conserving property, linear instability property, and the enhancement of high frequency noise. Some initial results of such measurements have already been reported for a one-dimensional model [8, 9]. An extensive experimental study of a two-dimensional model has also been reported [10].

The effect of discrete time step is not considered here and is believed to be quite small compared with the effect of a spatial grid.

Experimental Results

In the following, results of numerical simulations are shown together with comparisons witht the theory [7, 11]. First, a Maxwellian plasma is examined when the Debye length is small compared with the grid size. Then a case is followed when the Debye length is equal to the grid size. Finally, a one-beam plasma is examined showing a strong instability for all models when the beam is cold. The parameters of the experiments are listed in Table I.

TABLE I

Parameters of the Experiments

Experiment	Debye length λ_D	Model	Length of one period L	Thermal velocity v_t	Drift velocity v_d	Total no. of particles N
1	0.1	NGP	8	0.1	0	3168
2	0.1	CIC, PIC	8	0.1	0	2376
3	0.1	Dipole	16	0.1	0	3168
4	1.0	NGP	64	1.0	0	1000
5	1.0	CIC, PIC	64	1.0	0	1000
6	1.0	Dipole	64	1.0	0	1000
7	0	CIC, PIC	64	0	0.16	1000
8	0	Dipole	64	0	0.12	1000
9	0	Lewis	64	0	0.12	1000

time step of integration; $\omega_{pe}\Delta t = 0.2$
$\Delta x = 1$
$\omega_{pe} = 1$

A. Maxwellian plasma with a small Debye length

Experiment 1. NGP model was first checked using an ordered start [12]. 3168 particles (sheets) where arranged into 32 groups each with 99 particles. All groups have the same Maxwell velocity distribution. A small modulation in velocity space was added to the Maxwell distribution to excite the second mode ($\lambda = L/2$) which is the most unstable-mode in the model; with the uniform initial loading being noiseless, the initial modulation was found necessary.

Figure 1 shows the evolution of the instability in phase space at different time steps. The field, kinetic and total energies with the development of the instability are given in Fig. 2. A small modulation emerges at the early time steps, which is

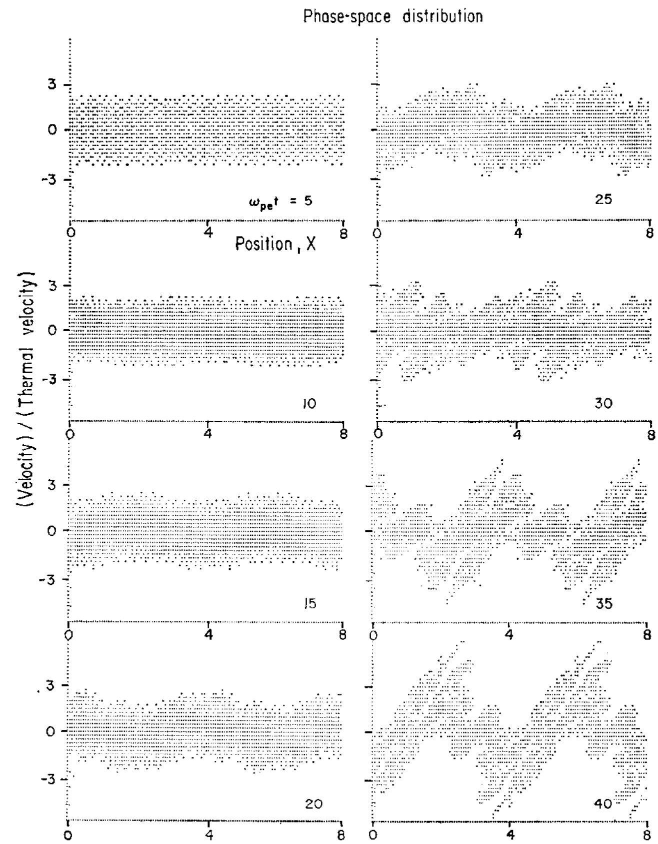

FIG. 1. Evolution of a grid instability in phase space for a Maxwellian plasma. A quiet start was used. $\lambda_D/\Delta x = 0.1$, $N_D = 40$, and $L/\Delta x = 8$. Experiment 1. NGP.

due to the initial condition. The growth rate ($\omega_i/\omega_{pe} = 0.1$) and the oscillation frequency ($\omega_r/\omega_{pe} = 0.78$) of the second mode are in good agreement with the linear theory [7]. The modulation continues to grow and becomes appreciable at $\omega_{pe}t = 20$, which is almost three plasma periods. At this stage, the field energy reached several percent of the kinetic energy and the total energy itself begins to

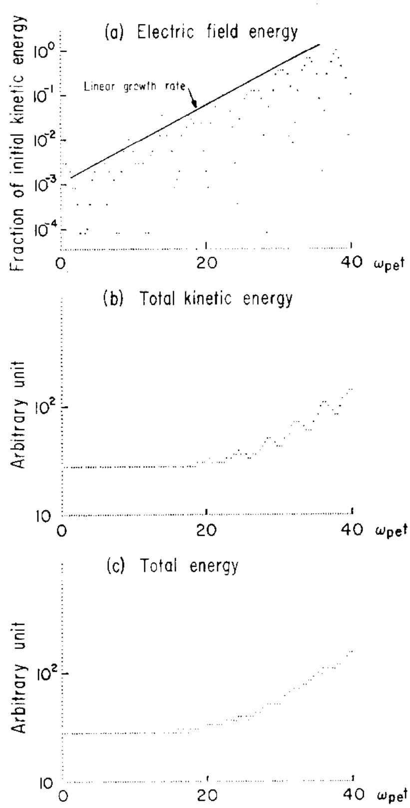

FIG. 2. Growth of the total field (a), kinetic (b) and the total (c) energies with time. Field energy normalized by the initial kinetic energy grows exponentially with the theoretically expected growth rate and the frequency as shown by the soild line. Experiment 1. NGP.

increase rapidly as shown in Fig. 2. After this stage, the energy continues to grow without appreciable saturation or limiting. Near the end of the calculation ($\omega_{pe}t = 40$), the kinetic energy reaches several times its initial value, resulting in the increase of the Debye length by a factor two and reduction of the growth rate. However, the catastrophe has already taken place. Higher spatial harmonics are generated with the development of the instability.

Experiment 2. Now we consider the case in which linear interpolation is used for charge and force assignment (CIC, PIC). The parameters of the experiments are the same as in Expt. 1 except for the total number of the particles, which is now reduced to 2376. Linear analysis shows that the maximum growth rate is $\omega_i/\omega_{pe} = 1.3 \times 10^{-2}$, which is one order of magnitude smaller than the NGP due to the smoothing of the interaction.

Figure 3 shows the development of the field energy of Fourier modes in the

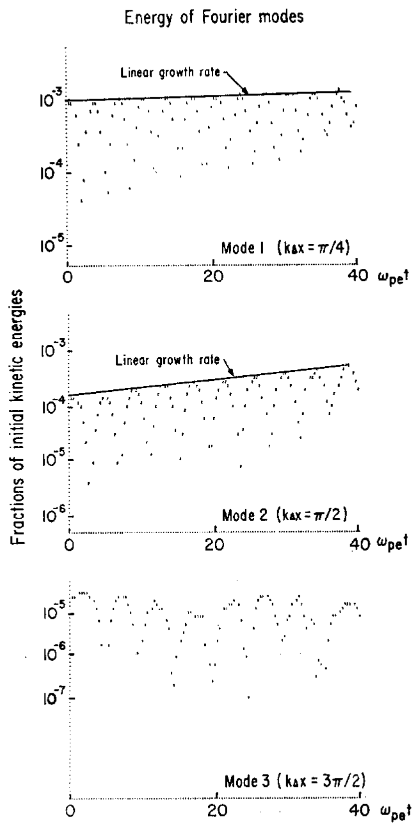

FIG. 3. The energies of the Fourier modes normalized by the kinetic energy. The growth rates are much smaller than those in Expt. 1 and are close to the theoretical predictions shown by the solid lines. $\lambda_D/\Delta x = 0.1$, $N_D = 30$, and $L/\Delta x = 8$. Experiment 2. CIC, PIC.

model. It is predicted that the first mode ($k\,\Delta\,x = \pi/4$) is weakly unstable ($\omega_i/\omega_{pe} = 10^{-3}$) and the second mode ($k\,\Delta\,x = \pi/2$) is at about the maximum growth rate ($\omega_i/\omega_{pe} = 1.3 \times 10^{-2}$), and the third mode will Landau-damp. The first and second modes grow with nearly the expected growth rate and the oscillation frequency. The total energy (not shown) was conserved within 0.5 % up to the end of calculation. Hence the growth rate of the instability is small, the maximum field energy was 0.25 % of the kinetic energy and this value is not large enough to lead the plasma to blow up. Fourier modes are still in the linear stage during the calculation. Although the instability is weak, it is expected that the effects of the instability will be appreciable for a longer run.

Experiment 3. We now consider an instability with the dipole expansion model. Figure 4 shows the development of the total field and kinetic energies for the same

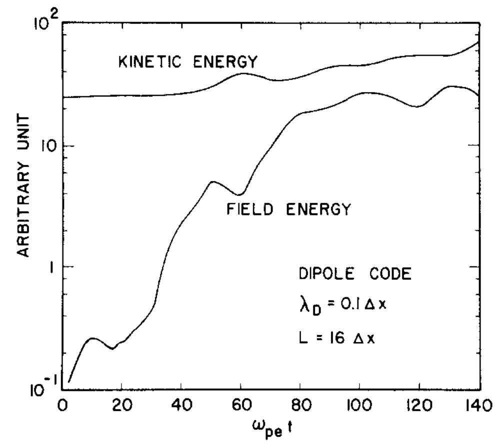

FIG. 4. Total field and kinetic energies of the Experiment 3 for the dipole expansion code. $\lambda_D/\Delta x = 0.1$, $L/\Delta x = 16$. Experiment 3. Dipole. Experiment 3. Dipole.

physical parameters as before. The field energy exponentiates up to until $\omega_{pe}t = 80$, where the kinetic energy begins to increase appreciably. Then the instability gradually saturates as the Debye length increases with the increase of the kinetic energy as Expt. 1. The phase-space structure is also similar to that of Expt. 1. The overall growth rate is estimated from Fig. 4 and it is about 0.3 of ω_{pe} and is greater than that of CIC, PIC models.

A similar experiment is performed with the linear version of Lewis energy conserving code in which charge density is calculated by using a linear interpolation, while the electric field is calculated from the nearest grid point method [4]. No instabilities were observed and the total energy was conserved one part in 10^5 which is consistent with the theory [7, 11].

These experiments are the cases where the maximum growth of the instability can be expected and, in fact, the instability becomes weaker by either increasing or decreasing the Debye length. Several additional experiments are performed for larger Debye lengths for practical interests. For the NGP model, cases where the Debye length is 0.2 and 0.3 of the grid size are examined. An appreciable reduction of the growth rate was immediately observed ($\omega_i/\omega_{pe} = 3.0 \times 10^{-2}$, 5.0×10^{-3} for $\lambda_D = 0.2$ and $0.3\ \Delta\ x$) and for $\lambda_D \gtrsim 0.5\ \Delta\ x$, no appreciable instabilities were observed. For CIC, PIC and dipole expansion codes, the growth rate was quickly reduced with the increase of the debye length and no instabilities were observed when the Debye length was larger than about $0.3\ \Delta\ x$.

Therefore, it is certain that there is a numerical instability for most of the codes when the Debye length is too small compared with the grid size. The instability is quite strong for the NGP model and for CIC, PIC and the dipole expansion codes, the instability will be weaker. It appears that one should not choose the grid size much larger than the Debye length to avoid a numerical instability and this is consistent with a physical discussion that the grid size gives the minimum resolution of the model and should be smaller than the Debye length.

For a large enough Debye length or a small enough grid, no linear instability

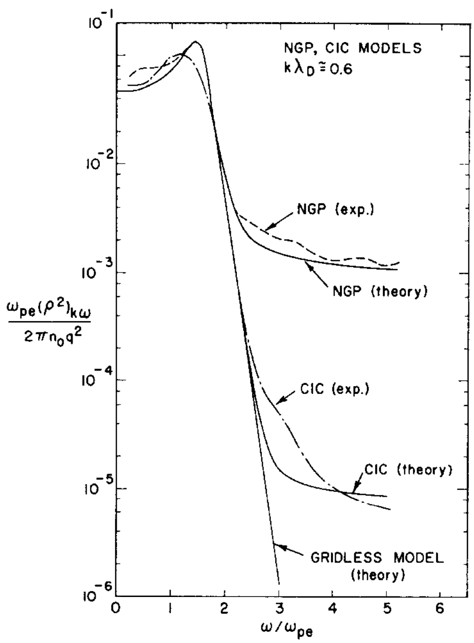

FIG. 5. Fluctuation spectrum of the charge density for the NGP and CIC models with the comparison with theory. For the NGP model, the enhanced noise is quite large. $\lambda_D/\Delta x = 1.$, $N_D = 16$ and $L/\Delta x = 64$. The wavenumber of the mode is $k\lambda_D = 0.6$. Experiments 4 and 5. NGP and CIC.

will be excited. However, the linear instability or the total energy conservation gives only a partial information of the behavior of a model. A more precise information about the behavior of a model can be obtained from a more careful measurement. The next section is devoted for the measurement of the fluctuation spectrum which includes the most detailed information.

B. *Maxwellian plasma with the Debye length equal to the grid size*

Experiments 4, 5, and 6. In this section, we show the measurements of the fluctuation spectrum of the charge density $(\rho^2)_{k\omega}$ in a Maxwellian plasma with the Debye length equal to the grid size; and hence no linear instability is expected. Theoretical predictions [11] are compared with the observation for NGP and CIC models. The measurements of the spectrum have been carried out by Fourier

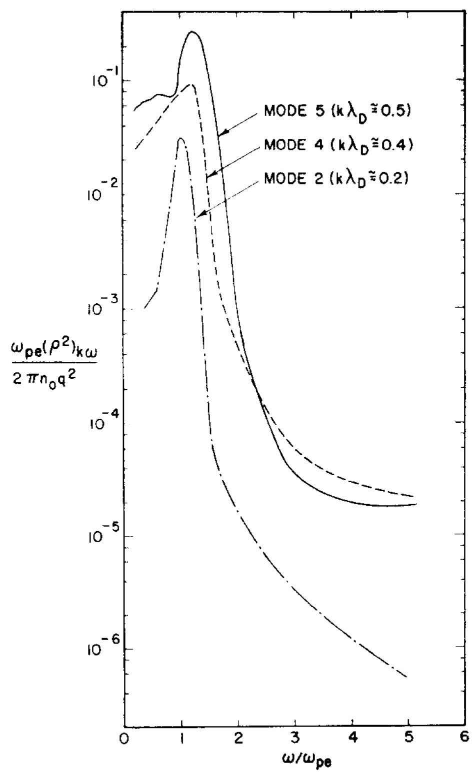

FIG. 6. Fluctuation spectrum of the charge density for the dipole code. It is observed that the dipole code and the CIC model show the same order of the noise. $\lambda_D/\Delta x = 1$, $N_D = 16$ and $L/\Delta x = 64$. Experiment 6. Dipole.

transforming the correlation function of the charge density using a standard technique [13].

Figure 5 shows the fluctuation spectrum for NGP (Expt. 4) and CIC (Expt. 5) models. The length of the system is 64 Debye length and the calculation was carried out until 200 $\omega_{pe}t$ for all the experiments shown in this section. The wave-number of the mode shown in Fig. 5 is $k\lambda_D = 0.6$. Enhanced high frequency noises can clearly be seen for both NGP and CIC models where no such high frequency oscillations exist in a real plasma or a gridless plasma. They agree quite well with the theoretical predictions [11]. The enhanced noise for the NGP model is not small at all and can play an appreciable role for physical processes. This noise is reduced two orders of magnitudes by using a smoother (linear) interpolations of charge and force and will be completely negligeble for CIC and PIC models for practical points of view. Several other modes with different wavelengths ($k\lambda_D = 0.1 \sim 1.$) were examined and it is confirmed that the enhancement of the noise is always appreciable for the NGP model even for a long wavelength mode ($k\lambda_D \ll 1.$). For CIC, PIC models, it is quite small and can be negligeble for most of the simulations. The total energy conservation was within 0.2 % for the CIC model and 0.6 % for the NGP model throughout the whole calculation.

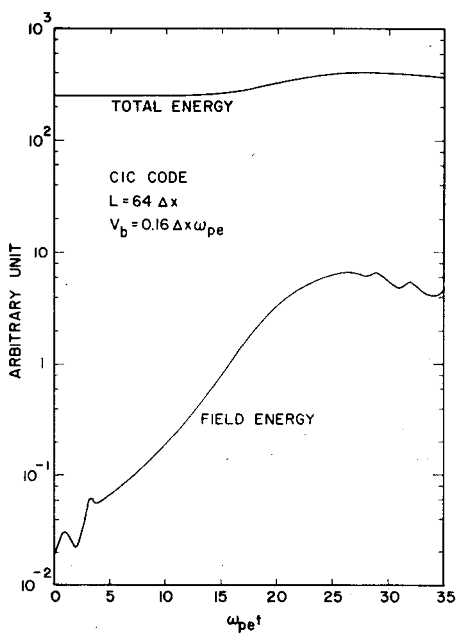

FIG. 7. One beam instability for the CIC model. The growth rate is about $0.14\omega_{pe}$. The beam is strictly cold. $v_d/\omega_{pe}\Delta x = 0.16$, $L/\Delta x = 64$. Experiment 7. CIC, PIC.

Figure 6 shows similar measurements for the dipole expansion model for three modes (Expt. 6). It is observed that the enhanced noise is much less than the NGP model and is about the same level as the CIC model. This fact was confirmed for the other modes in the model. It appears that the dipole expansion scheme is the same order of approximation as the linear interpolation scheme [5]. The conservation of the energy was within 0.1 % for this calculation.

C. *Instabilities in a one beam plasma*

Experiments 7, 8 *and* 9. Here we consider another class of numerical instability in a plasma beaming with respect to a spatial grid. This instability occurs when a plasma is streaming slowly with respect to a grid so that some of the grid aliased modes see the beam at their phase velocities. Therefore, in general, short wavelength modes whose wavelengths are comparable to the grid size are dominantly unstable.

Three experiments are shown in Figs. 7–9 (Expts. 7–9), when a beam is strictly

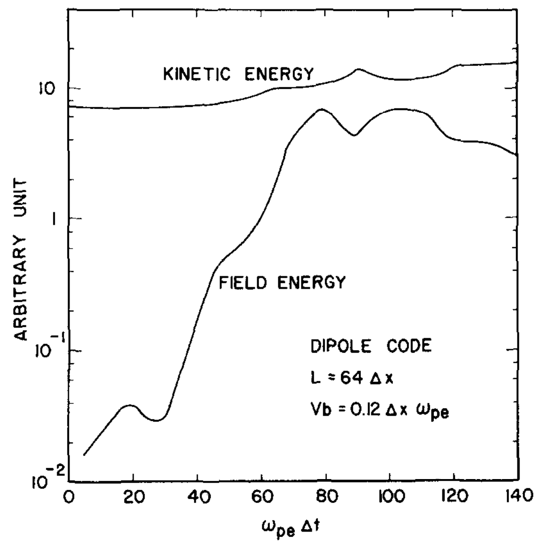

FIG. 8. One beam instability for the Dipole code. The growth rate is about $0.05\omega_{pe}$. $v_d/\omega_{pe}\Delta x = 0.12$, $L/\Delta x = 64$. Experiment 8. Dipole.

cold. The parameters of the experiments are summarized in Table I. For CIC (Fig. 7) and the dipole (Fig. 8) codes, the feature of the instability is very similar to a Maxwellian plasma shown in A. First the field energy grows exponentially and saturates at the time when the kinetic energy increases appreciably. The saturation of the instability is due to the increase of the Debye length, as before. The growth rate is large and more than 10 % and about 4 % of ω_{pe} for CIC and dipole expansion codes, respectively. The observed large growth rate for the CIC model is consistent with the prediction [11].

It is interesting to observe that Lewis energy conserving code also shows this instability as shown in Fig. 9. The total energy is conserved strictly in spite of the instability in this model. The growth rate is as large as $0.1\omega_{pe}$.

We looked at the case where the beam is not strictly cold adding a small thermal spread of the order of the beam velocity. The Debye length of the beam is therefore small and is about $0.1\ \Delta x$. No appreciable instability was observed for all of the models due to a small thermal spread in the beam velocity.

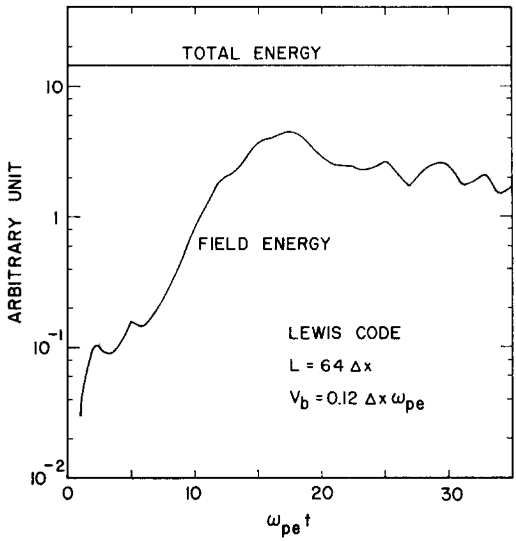

FIG. 9. One-beam instability for Lewis code. The growth rate is about $0.1\omega_{pe}$. The total energy is strictly conserved in spite of the instability. $v_d/\omega_{pe}\Delta x = 0.12$, $L/\Delta x = 64$. Experiment 9. Lewis code.

CONCLUSION

We have confirmed by numerical experiments that nonphysical instabilities can develop in a physically stable plasma due to the presence of grid aliased modes. When the Debye length is too small compared with the grid size, the instability has a large growth rate and can grow to large amplitudes, causing the total energy to increase. When the Debye length is about the same as the grid size, no instability was observed. However, the enhanced noise at high frequencies may not be negligeble for the NGP model.

It is confirmed that the dipole expansion model is much more quiet than the NGP model and appears to be the same order of approximation to CIC, PIC models.

Acknowledgment

The author is grateful to Dr. A. B. Langdon whose theoretical analysis is the motivation of this calculation. He is also indebted to Professors C. K. Birdsall and J. M. Dawson for useful comments.

This work was supported by the Atomic Energy Commission, contract AT(04-3) 34, P.A. 128 at the University of California, Berkeley and by the Office of the Naval Research Laboratory Contract N 00014-67-A-0151-0021 at the Plasma Physics Laboratory, Princeton University.

References

1. P. Burger, D. A. Dunn, and A. S. Halsted, *Phys. Fluids* **8** (1965), 2263; R. W. Hockney, *Phys. Fluids* **9** (1966), 1826.
2. C. K. Birdsall and D. Fuss, *J. Computational Phys.* **3** (1969), 494.
3. R. L. Morse and C. W. Nielson, *Phys. Fluids* **12** (1969), 2418. Also R. L. Morse, *in* "Methods in Computational Physics" (B. Alder, S. Fernbach, and M. Rotenberg, Eds.), Vol. 9, p. 213, Academic Press, New York, 1970.
4. H. R. Lewis, *J. Computational Phys.* **6** (1970), 136. Also H. R. Lewis *in* "Methods in Computational Physics" (B. Alder, S. Fernbach, and M. Rotenberg, Eds.) Vol. 9, p. 307, Academic Press, New York, 1970.
5. W. L. Kruer and J. M. Dawson, *Bull. Amer. Phys. Soc.* **14** (1969), 1025. Also W. L. Kruer and J. M. Dawson, paper presented at the Third Annual Numerical Plasma Simulation Conference, Stanford, CA, 1969.
6. E. L. Lindman, *J. Computational Phys.* **5** (1970), 13.
7. A. B. Langdon, *J. Computational Phys.* **6** (1970), 247.
8. H. Okuda, paper presented at the Fourth Conference on Numerical Simulation of Plasmas, Washington, DC, 1970.
9. A. Sykes, H. R. Lewis, and J. A. Wesson, *Bull. Amer. Phys. Soc.* **16** (1971), 1252.
10. R. W. Hockney, *J. Computational Phys.* **8** (1971), 19.
11. A. B. Langdon, to be published. Also A. B. Langdon, paper presented at the Fourth Conference on Numerical Simulation of Plasmas, Washington, DC, 1970.
12. J. A. Byers and M. Grewal, *Phys. Fluids* **13** (1970), 1819. Also J. A. Byers, paper presented at the Fourth Conference on Numerical Simulation of Plasmas, Washington, DC, 1970.
13. R. Blackman and J. W. Tukey, "The Measurement of Power Spectra," p. 12, Dover, New York, 1959.